Recursive LMMSE Filtering for Target Tracking with Range and Direction Cosine Measurements*

Zhansheng Duan Yu Liu X. Rong Li
Department of Electrical Engineering
University of New Orleans
New Orleans, LA 70148, U.S.A.
{zduan,lyu2,xli}@uno.edu

Abstract – *Due to the nonlinear relationship between Cartesian coordinates and range-direction-cosine coordinates, target tracking of state described in Cartesian coordinates with range and direction cosine measurements is a nonlinear filtering problem. Measurement conversion based Kalman filter available for this type of problem has some serious drawbacks. Depending on whether measurement of the third direction cosine is directly available, two recursive linear minimum mean-squared error (LMMSE) filters for target tracking with range and direction cosine measurements are developed in this paper. Illustrative numerical examples show that in terms of credibility and accuracy, the proposed filters should be preferred.*

Keywords: LMMSE, target tracking, range, direction cosine, nonlinear filtering.

1 Introduction

In tracking applications, target dynamics are usually modeled in Cartesian coordinates, while the measurements are directly available in the original sensor coordinates. Due to the nonlinear relationship between Cartesian coordinates and sensor coordinates, tracking in Cartesian coordinates using measurements available in sensor coordinates is in essence a nonlinear filtering problem. Existing nonlinear filtering algorithms, e.g., extended Kalman filter (EKF) [1], unscented filtering (UF) [2], the second-order Stirling interpolation based filter (DD2) [3], and Gaussian filter [4], for general nonlinear systems can be applied to this problem. But since the nonlinear relationship between the Cartesian coordinates and sensor coordinates is explicitly known, specifically designed nonlinear filters may behave better.

In a track-while-scan surveillance system, the radar can provide measurements, e.g., range, bearing, elevation and range rate, in polar or spherical coordinates [5]. Specifically designed target tracking algorithms for this type of radar abound. One popular idea is to convert the measurement model in polar or spherical coordinates to Cartesian coordinates so that the converted measurement model takes a pseudo linear form with respect to the target motion state described in Cartesian coordinates. In this way, it is hoped that the Kalman filter can be applied. Unfortunately, the converted measurement errors are state dependent and have biases. Numerous ways have been proposed to debias the converted measurements [5]. For instance, [6, 7, 8] proposed to use additive debiasing, and [9, 10] proposed to use multiplicative debiasing. [11] proposed the recursive LMMSE filter with respect to the converted measurements. [12, 13] extended debiased conversion to the case when range rate measurements are also available.

For the scan-while-track surveillance system, such as the phased array radar, the measurements are usually provided in RUV coordinates [5, 14, 15, 16, 17, 18, 19]. That is, what is available is range and two direction cosine measurements. Specifically designed target tracking algorithms for this type of radar are also available. For example, by using the first-order and second-order Taylor series expansions, [8, 20] developed Kalman filters for this type of problem based on debiased measurement conversion. As pointed out in [5, 11], the Kalman filter based on debaised conversion of measurement model from polar or spherical coordinates to Cartesian coordinates has the following serious drawbacks. First, the converted measurement errors are state dependent; second, their covariances are estimated conditioned on the measurement or state; third, the converted measurement error sequence is not white any more. However, in the assumptions of the Kalman filter, the measurement noise is independent of the state, its covariance is unconditional, and it is white. These drawbacks cannot be overcome in the existing debiased conversion based methods [6, 7, 8, 9, 10]. Since the debiased conversion based Kalman filters in [8, 20] follow the same idea, they are not free of these drawbacks either.

In this paper, we want to extend the work in [11] to target tracking with range and direction cosine measurements. That is, we want to develop a recursive LMMSE filter for target tracking in the RUV coordinates. It is shown that the main difficulty to fulfill this goal stems from the lack of direct measurement of the third direction cosine. For the

*Research supported in part by NSFC grant 60602026, ARO through grant W911NF-08-1-0409, ONR-DEPSCoR through grant N00014-09-1-1169, and NAVO through Contract # N62306-09-P-3S01.

case when the third direction cosine is measured, we propose a neat recursive LMMSE filter with range and direction cosine measurements; otherwise, by constructing a pseudo measurement of the third direction cosine, we also propose a slightly more complicated recursive LMMSE filter with range and two directly measured direction cosine measurements. Illustrative examples are provided to compare the proposed filters with some existing algorithms in terms of credibility and estimation accuracy.

The paper is organized as follows. Sec. 2 formulates the problem. Sec. 3 presents a recursive LMMSE filter when the third direction cosine measurement is directly available. Sec. 4 presents a recursive LMMSE filter when the third direction cosine measurement is not directly available. Sec. 5 provides some illustrative examples. Sec. 6 gives conclusions.

2 Problem formulation

To simplify discussion, consider only the following linear target dynamics described in Cartesian coordinates

$$X_k = F_{k-1}X_{k-1} + G_{k-1}W_{k-1}$$

where $X_k = [\ x_k \quad \dot{x}_k \quad y_k \quad \dot{y}_k \quad z_k \quad \dot{z}_k\ ]'$ is the target state vector at time k, $\langle W_k \rangle$ is a zero-mean white noise sequence with covariance $Q_k \geq 0$, $E[X_0] = \bar{X}_0$, $\text{cov}(X_0) = P_0$.

A radar located at the origin of the Cartesian coordinates measures range r_k^m, two direction cosines u_k^m and v_k^m of the target as

$$r_k^m = r_k + n_k^r$$
$$u_k^m = u_k + n_k^u \tag{1}$$
$$v_k^m = v_k + n_k^v \tag{2}$$

where

$$r_k = \sqrt{x_k^2 + y_k^2 + z_k^2},\ u_k = x_k/r_k,\ v_k = y_k/r_k$$

$\langle n_k^r \rangle$, $\langle n_k^u \rangle$ and $\langle n_k^v \rangle$ are all zero-mean white noise sequences with variances σ_r^2, σ_u^2 and σ_v^2, respectively; also, $\langle n_k^r \rangle$, $\langle n_k^u \rangle$ and $\langle n_k^v \rangle$ are independent of each other, and independent of X_0 and $\langle W_k \rangle$.

The range and two direction cosine measurements can be transformed into x_k^m and y_k^m in the Cartesian coordinates as

$$\begin{aligned} x_k^m &= r_k^m u_k^m = (r_k + n_k^r)(u_k + n_k^u) \\ &= x_k + r_k n_k^u + u_k n_k^r + n_k^r n_k^u \end{aligned} \tag{3}$$
$$\begin{aligned} y_k^m &= r_k^m v_k^m = (r_k + n_k^r)(v_k + n_k^v) \\ &= y_k + r_k n_k^v + v_k n_k^r + n_k^r n_k^v \end{aligned} \tag{4}$$

In this paper, we want to estimate the state as best as we can in the LMMSE sense.

Remark: Note that in the above formulation, we only need to know the first two moments of the measurement noise, while in [11, 6, 7, 8, 9, 10], distribution of the measurement noise must be known. For example, it was assumed therein that measurement noise is Gaussian distributed.

3 Recursive LMMSE filter with measured w_k^m

From the appendix of [11], it is easy to verify that the following two lemmas hold for LMMSE estimation.

Lemma 1 For a scalar-valued γ, if γ is uncorrelated with x and z, then

$$E^*[\gamma x|z] = E[\gamma]E^*[x|z]$$

Lemma 2 For LMMSE estimation error covariance,

$$\text{cov}(\tilde{x}) = E[(x-\hat{x})(x-\hat{x})'] = E[xx'] - E[\hat{x}\hat{x}']$$
$$\text{cov}(\tilde{x},\tilde{y}) = E[(x-\hat{x})(y-\hat{y})'] = E[xy'] - E[\hat{x}\hat{y}']$$

where

$$\hat{x} = E^*[x|z],\ \tilde{x} = x - \hat{x}$$
$$\hat{y} = E^*[y|z],\ \tilde{y} = y - \hat{y}$$

First, let us consider the case in which the third direction cosine measurement

$$w_k^m = w_k + n_k^w$$

is also available, where

$$w_k = z_k/r_k$$

and $\langle n_k^w \rangle$ is a zero-mean white noise sequence with variance σ_w^2. Also, $\langle n_k^w \rangle$ is statistically independent of $\langle n_k^r \rangle$, $\langle n_k^u \rangle$ and $\langle n_k^v \rangle$.

Correspondingly in the Cartesian coordinates, we have

$$\begin{aligned} z_k^m &= r_k^m w_k^m = (r_k + n_k^r)(w_k + n_k^w) \\ &= z_k + r_k n_k^w + w_k n_k^r + n_k^r n_k^w \end{aligned} \tag{5}$$

Denote

$$Z_k = [\ x_k^m \quad y_k^m \quad z_k^m\ ]'$$

Then by the two lemmas above, for the case in which all three direction cosine measurements u_k^m, v_k^m and w_k^m are directly available, the following theorem holds.

Theorem 1 (LMMSE filter with measured w_k^m). If the third direction cosine measurement w_k^m is also available, given $\hat{X}_{k-1|k-1} = E^*[X_{k-1}|Z^{k-1}]$, $P_{k-1|k-1} = \text{MSE}(\hat{X}_{k-1|k-1})$, the LMMSE estimate of X_k is:

Prediction:

$$\hat{X}_{k|k-1} = F_{k-1}\hat{X}_{k-1|k-1}$$
$$P_{k|k-1} = F_{k-1}P_{k-1|k-1}F'_{k-1} + G_{k-1}Q_{k-1}G'_{k-1}$$

Update:

$$\hat{X}_{k|k} = \hat{X}_{k|k-1} + C_{k|k-1}S_k^{-1}\tilde{Z}_{k|k-1}$$
$$P_{k|k} = P_{k|k-1} - C_{k|k-1}S_k^{-1}C'_{k|k-1}$$

where

$$
\begin{aligned}
\tilde{Z}_{k|k-1} &= Z_k - \hat{Z}_{k|k-1} \\
\hat{Z}_{k|k-1} &= [\hat{X}_{k|k-1}(1), \hat{X}_{k|k-1}(3), \hat{X}_{k|k-1}(5)]' \\
C_{k|k-1} &= [P_{k|k-1}(:,1), P_{k|k-1}(:,3), P_{k|k-1}(:,5)] \\
S_k &= [S_k(i,j)]_{i,j=1}^3 \\
S_k(1,1) &= P_{k|k-1}(1,1) + \sigma_u^2 E[r_k^2] + \sigma_r^2 E[u_k^2] + \sigma_r^2\sigma_u^2 \\
S_k(1,2) &= S_k(2,1) = P_{k|k-1}(1,3) + \sigma_r^2 E[u_k v_k] \\
S_k(1,3) &= S_k(3,1) = P_{k|k-1}(1,5) + \sigma_r^2 E[u_k w_k] \\
S_k(2,2) &= P_{k|k-1}(3,3) + \sigma_v^2 E[r_k^2] + \sigma_r^2 E[v_k^2] + \sigma_r^2\sigma_v^2 \\
S_k(2,3) &= S_k(3,2) = P_{k|k-1}(3,5) + \sigma_r^2 E[v_k w_k] \\
S_k(3,3) &= P_{k|k-1}(5,5) + \sigma_w^2 E[r_k^2] + \sigma_r^2 E[w_k^2] + \sigma_r^2\sigma_w^2
\end{aligned}
$$

Proof:

Given $\hat{X}_{k-1|k-1}$ and $P_{k-1|k-1}$, it follows easily from the property of LMMSE estimation that

$$
\begin{aligned}
\hat{X}_{k|k-1} &= E^*[X_k|Z^{k-1}] = F_{k-1}\hat{X}_{k-1|k-1} \\
P_{k|k-1} &= \text{MSE}(\hat{X}_{k|k-1}) \\
&= F_{k-1}P_{k-1|k-1}F'_{k-1} + G_{k-1}Q_{k-1}G'_{k-1}
\end{aligned}
$$

The LMMSE estimator $E^*[X_k|Z^k]$ always has the following *quasi-recursive* form [21]

$$
\begin{aligned}
\hat{X}_{k|k} &= E^*[X_k|Z^k] = E^*[X_k|Z^{k-1}, Z_k] \\
&= \hat{X}_{k|k-1} + C_{k|k-1}S_k^{-1}\tilde{Z}_{k|k-1} \\
P_{k|k} &= \text{MSE}(\hat{X}_{k|k}) = P_{k|k-1} - C_{k|k-1}S_k^{-1}C'_{k|k-1}
\end{aligned}
$$

From Lemma 1 and the property of LMMSE estimation, we have

$$
\begin{aligned}
\hat{Z}_{k|k-1} &= E^*[Z_k|Z^{k-1}] \\
&= [E^*[x_k^m|Z^{k-1}], E^*[y_k^m|Z^{k-1}], E^*[z_k^m|Z^{k-1}]]'
\end{aligned}
$$

where

$$
\begin{aligned}
E^*[x_k^m|Z^{k-1}] &= E^*[x_k + r_k n_k^u + u_k n_k^r + n_k^r n_k^u|Z^{k-1}] \\
&= E^*[x_k|Z^{k-1}] = \hat{x}_{k|k-1} = \hat{X}_{k|k-1}(1)
\end{aligned}
$$

and similarly,

$$
\begin{aligned}
E^*[y_k^m|Z^{k-1}] &= \hat{y}_{k|k-1} = \hat{X}_{k|k-1}(3) \\
E^*[z_k^m|Z^{k-1}] &= \hat{z}_{k|k-1} = \hat{X}_{k|k-1}(5)
\end{aligned}
$$

From Lemma 2, we have

$$
\begin{aligned}
&C_{k|k-1} \\
&= \text{cov}(\tilde{X}_{k|k-1}, \tilde{Z}_{k|k-1}) = E[X_k Z_k'] - E[\hat{X}_{k|k-1}\hat{Z}'_{k|k-1}] \\
&= E\begin{bmatrix} x_k x_k^m & x_k y_k^m & x_k z_k^m \\ \dot{x}_k x_k^m & \dot{x}_k y_k^m & \dot{x}_k z_k^m \\ y_k x_k^m & y_k y_k^m & y_k z_k^m \\ \dot{y}_k x_k^m & \dot{y}_k y_k^m & \dot{y}_k z_k^m \\ z_k x_k^m & z_k y_k^m & z_k z_k^m \\ \dot{z}_k x_k^m & \dot{z}_k y_k^m & \dot{z}_k z_k^m \end{bmatrix} \\
&- E\begin{bmatrix} \hat{x}_{k|k-1}\hat{x}_{k|k-1} & \hat{x}_{k|k-1}\hat{y}_{k|k-1} & \hat{x}_{k|k-1}\hat{z}_{k|k-1} \\ \widehat{\dot{x}}_{k|k-1}\hat{x}_{k|k-1} & \widehat{\dot{x}}_{k|k-1}\hat{y}_{k|k-1} & \widehat{\dot{x}}_{k|k-1}\hat{z}_{k|k-1} \\ \hat{y}_{k|k-1}\hat{x}_{k|k-1} & \hat{y}_{k|k-1}\hat{y}_{k|k-1} & \hat{y}_{k|k-1}\hat{z}_{k|k-1} \\ \widehat{\dot{y}}_{k|k-1}\hat{x}_{k|k-1} & \widehat{\dot{y}}_{k|k-1}\hat{y}_{k|k-1} & \widehat{\dot{y}}_{k|k-1}\hat{z}_{k|k-1} \\ \hat{z}_{k|k-1}\hat{x}_{k|k-1} & \hat{z}_{k|k-1}\hat{y}_{k|k-1} & \hat{z}_{k|k-1}\hat{z}_{k|k-1} \\ \widehat{\dot{z}}_{k|k-1}\hat{x}_{k|k-1} & \widehat{\dot{z}}_{k|k-1}\hat{y}_{k|k-1} & \widehat{\dot{z}}_{k|k-1}\hat{z}_{k|k-1} \end{bmatrix}
\end{aligned}
$$

where

$$E[x_k x_k^m] = E[x_k(x_k + r_k n_k^u + u_k n_k^r + n_k^r n_k^u)] = E[x_k x_k]$$

and similarly,

$$
\begin{aligned}
&E[x_k y_k^m] = E[x_k y_k],\ E[x_k z_k^m] = E[x_k z_k] \\
&E[\dot{x}_k x_k^m] = E[\dot{x}_k x_k],\ E[\dot{x}_k y_k^m] = E[\dot{x}_k y_k],\ E[\dot{x}_k z_k^m] = E[\dot{x}_k z_k] \\
&E[y_k x_k^m] = E[y_k x_k],\ E[y_k y_k^m] = E[y_k y_k],\ E[y_k z_k^m] = E[y_k z_k] \\
&E[\dot{y}_k x_k^m] = E[\dot{y}_k x_k],\ E[\dot{y}_k y_k^m] = E[\dot{y}_k y_k],\ E[\dot{y}_k z_k^m] = E[\dot{y}_k z_k] \\
&E[z_k x_k^m] = E[z_k x_k],\ E[z_k y_k^m] = E[z_k y_k],\ E[z_k z_k^m] = E[z_k z_k] \\
&E[\dot{z}_k x_k^m] = E[\dot{z}_k x_k],\ E[\dot{z}_k y_k^m] = E[\dot{z}_k y_k],\ E[\dot{z}_k z_k^m] = E[\dot{z}_k z_k]
\end{aligned}
$$

Thus

$$C_{k|k-1} = [P_{k|k-1}(:,1), P_{k|k-1}(:,3), P_{k|k-1}(:,5)]$$

Furthermore, from Lemma 2, we have

$$
\begin{aligned}
S_k &= \text{cov}(\tilde{Z}_{k|k-1}) = E[Z_k Z_k'] - E[\hat{Z}_{k|k-1}\hat{Z}'_{k|k-1}] \\
&= E\begin{bmatrix} x_k^m x_k^m & x_k^m y_k^m & x_k^m z_k^m \\ y_k^m x_k^m & y_k^m y_k^m & y_k^m z_k^m \\ z_k^m x_k^m & z_k^m y_k^m & z_k^m z_k^m \end{bmatrix} \\
&- E\begin{bmatrix} \hat{x}_{k|k-1}\hat{x}_{k|k-1} & \hat{x}_{k|k-1}\hat{y}_{k|k-1} & \hat{x}_{k|k-1}\hat{z}_{k|k-1} \\ \hat{y}_{k|k-1}\hat{x}_{k|k-1} & \hat{y}_{k|k-1}\hat{y}_{k|k-1} & \hat{y}_{k|k-1}\hat{z}_{k|k-1} \\ \hat{z}_{k|k-1}\hat{x}_{k|k-1} & \hat{z}_{k|k-1}\hat{y}_{k|k-1} & \hat{z}_{k|k-1}\hat{z}_{k|k-1} \end{bmatrix}
\end{aligned}
$$

where

$$
\begin{aligned}
&E[x_k^m x_k^m] \\
&= E[(x_k + r_k n_k^u + u_k n_k^r + n_k^r n_k^u)(\cdot)] \\
&= E[x_k x_k] + E[r_k^2\,(n_k^u)^2] + E[u_k^2\,(n_k^r)^2] + E[(n_k^r)^2\,(n_k^u)^2] \\
&= E[x_k x_k] + \sigma_u^2 E[r_k^2] + \sigma_r^2 E[u_k^2] + \sigma_r^2\sigma_u^2
\end{aligned}
$$

and similarly,

$$
\begin{aligned}
E[x_k^m y_k^m] &= E[y_k^m x_k^m] = E[x_k y_k] + \sigma_r^2 E[u_k v_k] \\
E[x_k^m z_k^m] &= E[z_k^m x_k^m] = E[x_k z_k] + \sigma_r^2 E[u_k w_k] \\
E[y_k^m y_k^m] &= E[y_k y_k] + \sigma_v^2 E[r_k^2] + \sigma_r^2 E[v_k^2] + \sigma_r^2\sigma_v^2 \\
E[y_k^m z_k^m] &= E[z_k^m y_k^m] = E[y_k z_k] + \sigma_r^2 E[v_k w_k] \\
E[z_k^m z_k^m] &= E[z_k z_k] + \sigma_w^2 E[r_k^2] + \sigma_r^2 E[w_k^2] + \sigma_r^2\sigma_w^2
\end{aligned}
$$

Thus

$$\begin{aligned}
S_k(1,1) &= P_{k|k-1}(1,1) + \sigma_u^2 E[r_k^2] + \sigma_r^2 E[u_k^2] + \sigma_r^2\sigma_u^2 \\
S_k(1,2) &= S_k(2,1) = P_{k|k-1}(1,3) + \sigma_r^2 E[u_k v_k] \\
S_k(1,3) &= S_k(3,1) = P_{k|k-1}(1,5) + \sigma_r^2 E[u_k w_k] \\
S_k(2,2) &= P_{k|k-1}(3,3) + \sigma_v^2 E[r_k^2] + \sigma_r^2 E[v_k^2] + \sigma_r^2\sigma_v^2 \\
S_k(2,3) &= S_k(3,2) = P_{k|k-1}(3,5) + \sigma_r^2 E[v_k w_k] \\
S_k(3,3) &= P_{k|k-1}(5,5) + \sigma_w^2 E[r_k^2] + \sigma_r^2 E[w_k^2] + \sigma_r^2\sigma_w^2
\end{aligned}$$

□

Remark: Since r_k, u_k, v_k and w_k are all nonlinear functions of X_k, the expectations $E[r_k^2]$, $E[u_k^2]$, $E[v_k^2]$, $E[w_k^2]$, $E[u_k v_k]$, $E[u_k w_k]$ and $E[v_k w_k]$ in Theorem 1 can be approximated by unscented transformation (UT). Only the UT for $E[r_k^2]$ is shown below for illustration, and the others can be done similarly.

$$\begin{aligned}
E[r_k^2] &= E[x_k^2 + y_k^2 + z_k^2] \\
&\approx \sum_{j=-6}^{6} \alpha^{(j)} ((\mathcal{X}_k^{(j)}(1))^2 + (\mathcal{X}_k^{(j)}(3))^2 + (\mathcal{X}_k^{(j)}(5))^2)
\end{aligned}$$

where

$$\begin{aligned}
&\mathcal{X}_k^{(0)} = \hat{X}_{k|k-1},\ \mathcal{X}_k^{(\pm i)} = \hat{X}_{k|k-1} \pm [((6+\kappa)P_{k|k-1})^{\frac{1}{2}}]_i \\
&\alpha^{(0)} = \frac{\kappa}{6+\kappa},\ \alpha^{(\pm i)} = \frac{1}{2(6+\kappa)},\ i = 1, 2, \cdots, 6
\end{aligned}$$

$[A^{\frac{1}{2}}]_i$ is the i-th column of the Cholesky decomposition of square matrix A. The design parameter κ provides an extra degree of freedom to "fine tune" the higher order moments of the approximation and κ can be any number (positive or negative) provided that $6 + \kappa \neq 0$.

4 Recursive LMMSE filter without measured w_k^m

As described in the problem formulation part, w_k^m is usually not measured in the current radar. This makes the corresponding LMMSE estimation problem much harder.

One popular way is to construct a pseudo measurement by

$$\begin{aligned}
w_k^m &= \sqrt{1 - (u_k^m)^2 - (v_k^m)^2} \qquad (6) \\
&= \sqrt{1 - (u_k + n_k^u)^2 - (v_k + n_k^v)^2}
\end{aligned}$$

Then the range and direction cosine measurements can be transformed to x_k^m, y_k^m and z_k^m in the Cartesian coordinates, as in (3), (4) and (5), with w_k^m replaced by (6) so that

$$\begin{aligned}
z_k^m = r_k^m w_k^m &= (r_k + n_k^r)\sqrt{1 - (u_k + n_k^u)^2 - (v_k + n_k^v)^2} \\
&= r_k\sqrt{1 - (u_k + n_k^u)^2 - (v_k + n_k^v)^2} \\
&\quad + n_k^r\sqrt{1 - (u_k + n_k^u)^2 - (v_k + n_k^v)^2}
\end{aligned}$$

Denote

$$Z_k = [\ x_k^m \quad y_k^m \quad z_k^m\ ]'$$

Then for the case without measured w_k^m, the following theorem holds.

Theorem 2 (LMMSE filter without measured w_k^m). If the third direction cosine measurement w_k^m is not available, given $\hat{X}_{k-1|k-1} = E^*[X_{k-1}|Z^{k-1}]$, $P_{k-1|k-1} = \text{MSE}(\hat{X}_{k-1|k-1})$, the LMMSE estimator of X_k is:

Prediction: Same as in LMMSE filter with measured w_k^m.

Update: Same as in LMMSE filter with measured w_k^m except that

$$\begin{aligned}
\hat{Z}_{k|k-1} &= [\hat{X}_{k|k-1}(1), \hat{X}_{k|k-1}(3), \hat{z}_{k|k-1}^m]' \\
C_{k|k-1} &= [P_{k|k-1}(:,1), P_{k|k-1}(:,3), C_{k|k-1}^{(\cdot 3)}] \\
S_k(\,:,3) &= S_k(3,:)' = S_k^{(\cdot 3)} \\
\hat{z}_{k|k-1}^m &= E^*[r_k\sqrt{1 - (u_k + n_k^u)^2 - (v_k + n_k^v)^2}|Z^{k-1}] \\
C_{k|k-1}^{(\cdot 3)} &= E[(X_k - \hat{X}_{k|k-1})(z_k^m - \hat{z}_{k|k-1}^m)] \\
S_k^{(\cdot 3)} &= E[(Z_k - \hat{Z}_{k|k-1})(z_k^m - \hat{z}_{k|k-1}^m)]
\end{aligned}$$

Proof:

From Lemma 1 and the property of LMMSE estimation, we have

$$\begin{aligned}
\hat{Z}_{k|k-1}(3) &= E^*[z_k^m|Z^{k-1}] \triangleq \hat{z}_{k|k-1}^m \\
&= E^*[r_k\sqrt{1 - (u_k + n_k^u)^2 - (v_k + n_k^v)^2} \\
&\quad + n_k^r\sqrt{1 - (u_k + n_k^u)^2 - (v_k + n_k^v)^2}|Z^{k-1}] \\
&= E^*[r_k\sqrt{1 - (u_k + n_k^u)^2 - (v_k + n_k^v)^2}|Z^{k-1}]
\end{aligned}$$

From Lemma 2, we have

$$\begin{aligned}
C_{k|k-1}(\,:,3) &= \text{cov}[\tilde{X}_{k|k-1}, \tilde{z}_{k|k-1}^m] \triangleq C_{k|k-1}^{(\cdot 3)} \\
&= E[(X_k - \hat{X}_{k|k-1})(z_k^m - \hat{z}_{k|k-1}^m)] \\
S_k(\,:,3) = S_k(3,:)' &= \text{cov}[\tilde{Z}_{k|k-1}, \tilde{z}_{k|k-1}^m] \triangleq S_k^{(\cdot 3)} \\
&= E[(Z_k - \hat{Z}_{k|k-1})(z_k^m - \hat{z}_{k|k-1}^m)]
\end{aligned}$$

□

Remark: In Theorem 2, $E[r_k^2]$, $E[u_k^2]$, $E[v_k^2]$ and $E[u_k v_k]$ can be approximated in the same way as in Theorem 1.

Remark: Since w_k^m is not directly measured in this case, $\hat{z}_{k|k-1}^m$ does not have a nice form as in Theorem 1. It can be approximated by some moment matching techniques [22], e.g., UT, as

$$\begin{aligned}
\hat{z}_{k|k-1}^m &= E^*[r_k\sqrt{1 - (u_k + n_k^u)^2 - (v_k + n_k^v)^2}|Z^{k-1}] \\
&\approx \sum\nolimits_{j=-9}^{9} \beta^{(j)} z_k^{(j)}
\end{aligned}$$

where

$$
\begin{aligned}
z_k^{(j)} &= r_k^{(j)}\sqrt{1-(u_k^{(j)}+\mathcal{Y}_k^{(j)}(8))^2-(v_k^{(j)}+\mathcal{Y}_k^{(j)}(9))^2}\\
r_k^{(j)} &= \sqrt{(\mathcal{Y}_k^{(j)}(1))^2+(\mathcal{Y}_k^{(j)}(3))^2+(\mathcal{Y}_k^{(j)}(5))^2}\\
u_k^{(j)} &= \mathcal{Y}_k^{(j)}(1)/r_k^{(j)}\\
v_k^{(j)} &= \mathcal{Y}_k^{(j)}(3)/r_k^{(j)},\ j=\pm 9,\pm 8,\cdots,0\\
\beta^{(0)} &= \frac{\kappa}{9+\kappa},\ \beta^{(\pm i)}=\frac{1}{2(9+\kappa)},\ \mathcal{Y}_k^{(0)}=Y_k\\
\mathcal{Y}_k^{(\pm i)} &= Y_k \pm [((9+\kappa)C_k)^{\frac{1}{2}}]_i,\ i=1,2,\cdots,9\\
Y_k &= [\hat{X}'_{k|k-1},0,0,0]',\ C_k=\text{diag}(P_{k|k-1},\sigma_r^2,\sigma_u^2,\sigma_v^2)
\end{aligned}
$$

And correspondingly,

$$
\begin{aligned}
C_{k|k-1}^{(\cdot 3)} &\approx \sum_{j=-9}^{9}\beta^{(j)}(\mathcal{Y}_k^{j}(1:6)-\hat{X}_{k|k-1})(z_k^{(j)}-\hat{z}_{k|k-1}^m)\\
S_k^{(\cdot 3)} &\approx \sum\nolimits_{j=-9}^{9}\beta^{j}(\mathcal{Z}_k^{(j)}-\hat{Z}_{k|k-1})(z_k^{(j)}-\hat{z}_{k|k-1}^m)
\end{aligned}
$$

where

$$
\begin{aligned}
\mathcal{Z}_k^{(j)} &= [\ x_k^{(j)}\quad y_k^{(j)}\quad z_k^{(j)}\ ]'\\
x_k^{(j)} &= \mathcal{Y}_k^{(j)}(1)+r_k^{(j)}\mathcal{Y}_k^{(j)}(8)+u_k^{(j)}\mathcal{Y}_k^{(j)}(7)\\
&\quad +\mathcal{Y}_k^{(j)}(7)\mathcal{Y}_k^{(j)}(8)\\
y_k^{(j)} &= \mathcal{Y}_k^{(j)}(3)+r_k^{(j)}\mathcal{Y}_k^{(j)}(9)+v_k^{(j)}\mathcal{Y}_k^{(j)}(7)\\
&\quad +\mathcal{Y}_k^{(j)}(7)\mathcal{Y}_k^{(j)}(9)
\end{aligned}
$$

Remark: Other moment matching techniques [22], e.g., DD2 [3], can also be used to replace UT.

5 Illustrative examples

In this section, we compare performance of the proposed recursive LMMSE filter with measured w_k^m (LMMSEw), the proposed recursive LMMSE filter without measured w_k^m (LMMSE), UF, converted measurement Kalman filter with CM1 (CM1KF) [20] and converted measurement Kalman filter with CM2 (CM2KF) [20] in terms of estimation accuracy and credibility through numerical examples. Except LMMSEw, all the other filters are assumed to have no access to the measured w_k^m.

Consider the following three-dimensional discrete-time constant velocity (CV) motion model [23] of a target

$$X_k = F_{k-1}X_{k-1}+G_{k-1}W_{k-1}$$

where

$$
\begin{aligned}
X_k &= [\ x_k\quad \dot{x}_k\quad y_k\quad \dot{y}_k\quad z_k\quad \dot{z}_k\ ]',\ X_0\sim\mathcal{N}(\bar{X}_0,P_0)\\
F_k &= \text{diag}(\mathcal{F},\mathcal{F},\mathcal{F}),\ G_k=\text{diag}(\mathcal{G},\mathcal{G},\mathcal{G})\\
\mathcal{F} &= \begin{bmatrix} 1 & T\\ 0 & 1\end{bmatrix},\ \mathcal{G}=\begin{bmatrix} T^2/2\\ T\end{bmatrix},\ T=10s\\
W_k &\sim \mathcal{N}([\ 0\quad 0\quad 0\ ]',\text{diag}((q_k^x)^2,(q_k^y)^2,(q_k^z)^2))
\end{aligned}
$$

The estimation accuracy measures used are root mean-squared (RMS) position and velocity errors. The filter credibility measures used are average normalized estimation error squared (ANEES) and noncredibility index (NCI) [24].

All results below are averaged over 150 Monte Carlo runs. All filters are initialized with

$$\hat{X}_{0|0}=\bar{X}_0,\ P_{0|0}=P_0$$

and

$$
\begin{aligned}
\bar{X}_0 &= [\ 3300km\quad 600m/s\quad 3300km\quad 600m/s\\
&\qquad 3300km\quad 600m/s\ ]'\\
P_0 &= \text{diag}(10^4m^2,100m^2/s^2,10^4m^2,100m^2/s^2,\\
&\qquad 10^4m^2,100m^2/s^2)\\
\sigma_u &= \sigma_v=\sigma_w=10^{-3},\ q_k^x=q_k^y=q_k^z=0.05m/s^2
\end{aligned}
$$

Figs. 1 through 4 show comparison results for the case with $\sigma_r = 100m$ (**case 1**). In this case, the range measurement accuracy is the poorest among all three cases. From the simulation results, it can be seen that in terms of both RMS position and velocity errors, there is no significant difference among CM1KF, CM2KF, UF and LMMSE. LMMSEw has the smallest RMS position error. In terms of RMS velocity error, LMMSEw outperforms all the others during steady state but is worse than all the others during transient. It terms of filter credibility, all filters are credible.

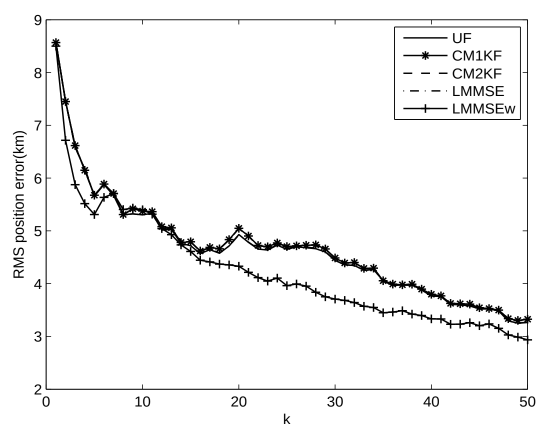

Figure 1: RMS position error comparison for case 1. Note that UF, CM1KF, CM2KF and LMMSE essentially overlap with each other except that CM1KF is slightly worse than the other three.

Figs. 5 through 8 show comparison results for the case with $\sigma_r = 10m$ (**case 2**). In this case, the range measurement accuracy is relatively better. From the simulation results, it can be seen that in terms of estimation accuracy, CM1KF is the worst except it beats LMMSEw in velocity slightly during transient. Our LMMSE beats CM2KF clearly but beats UF only slightly. In terms of RMS velocity error, there is no significant difference among CM2KF,

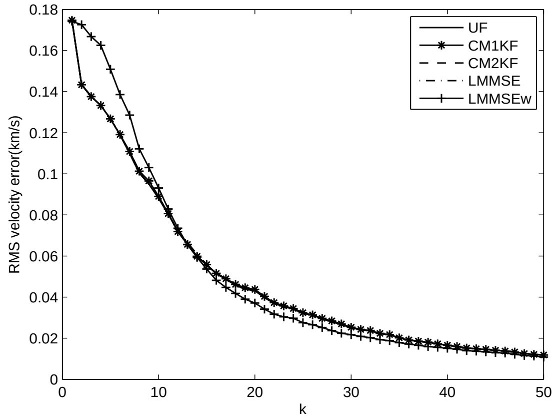

Figure 2: RMS velocity error comparison for case 1. Note that UF, CM1KF, CM2KF and LMMSE essentially overlap with each other.

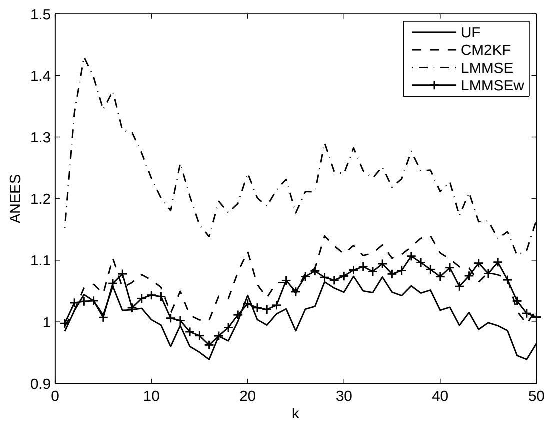

Figure 3: ANEES comparison for case 1

UF and LMMSE. Our LMMSEw beats all the others significantly in RMS position error. In terms of filter credibility, CM1KF is far from credible. (The ANEES and NCI of CM1KF are not shown because they are much larger than those of the other filters.) CM2KF is barely credible and all the others are credible.

Figs. 9 through 12 show comparison results for the case with $\sigma_r = 1m$ (**case 3**). In this case, the range measurement accuracy is the best. From the simulation results, it can be seen that in terms of estimation accuracy, the difference between CM1KF and all the others is increased a lot when compared with case 2.

Over all three cases, it can be seen that in terms of both estimation accuracy and filter credibility, LMMSEw beats all the others. Considering both this performance improvement and simplicity of the recursive LMMSE filter with measured w_k^m, it is highly desirable to have w_k^m in the same way (signal processing mechanism) as u_k^m and v_k^m in (1) and (2) instead of using the pseudo measurement in (6) in the phased array radar. Although our LMMSE has very close estimation accuracy to those of CM2KF and UF, its filter credibility is much better.

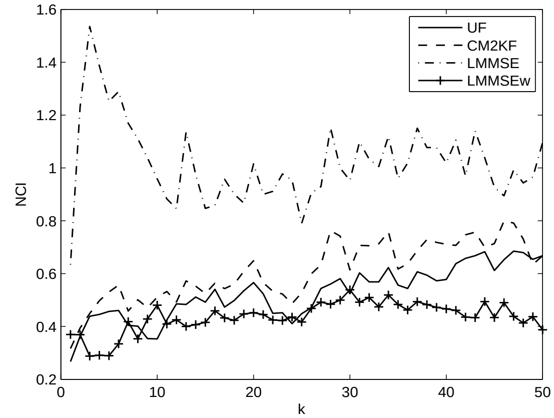

Figure 4: NCI comparison for case 1

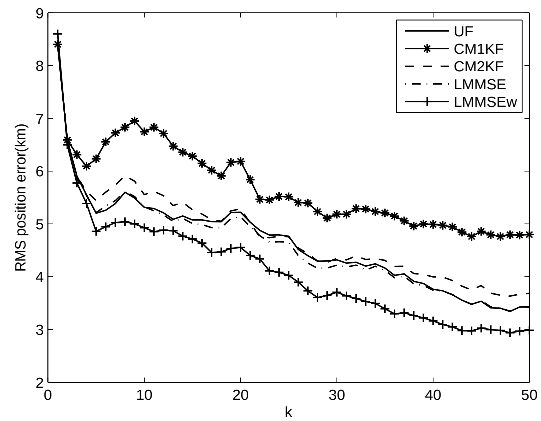

Figure 5: RMS position error comparison for case 2. Note that UF and LMMSE essentially overlap with each other except that LMMSE is slightly better than UF.

6 Conclusions

For a scan-while-track surveillance system, such as a phased array radar, the available measurements are usually range and direction cosine. To track the state of a moving target described in the Cartesian coordinates, one existing approach is to convert the range and direction cosine measurement model to Cartesian coordinates so that the converted measurement model takes a pseudo linear form with respect to the state, but it has some serious drawbacks. The main difficulty to develop a recursive LMMSE filter for this type of problem stems from the lack of direct measurement of the third direction cosine. For the case when the third direction cosine can be directly measured, we have proposed a neat recursive LMMSE filter; otherwise, by constructing a pseudo measurement of the third direction cosine, we have also proposed a slightly more complicated recursive LMMSE filter.

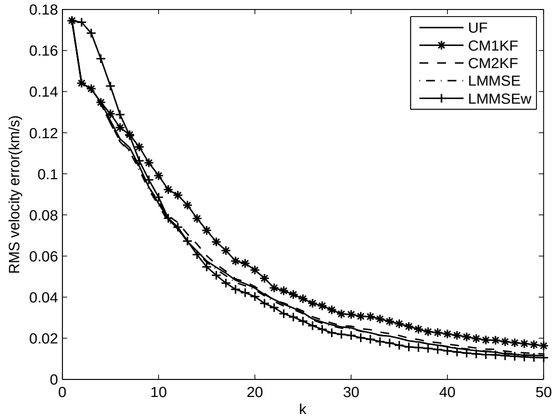

Figure 6: RMS velocity error comparison for case 2. Note that UF, CM2KF and LMMSE essentially overlap with each other except that UF is slightly better than CM2KF and LMMSE is slightly better than UF.

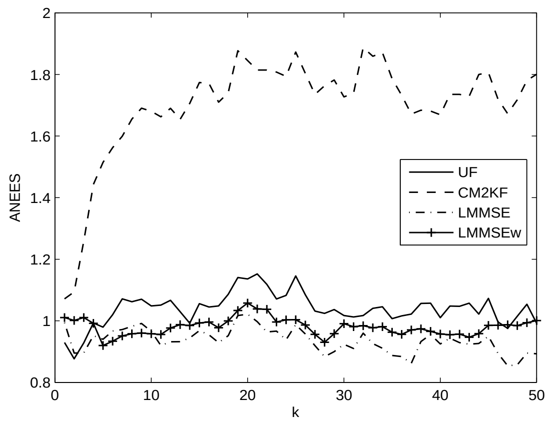

Figure 7: ANEES comparison for case 2

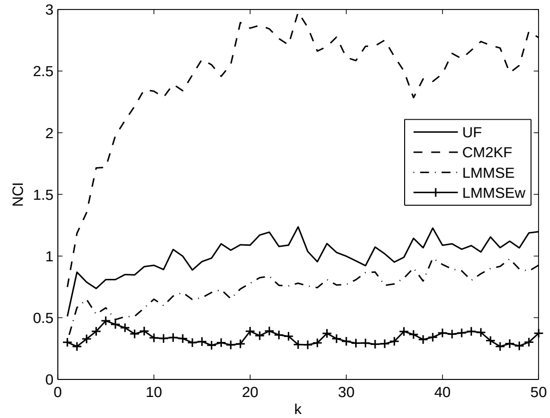

Figure 8: NCI comparison for case 2

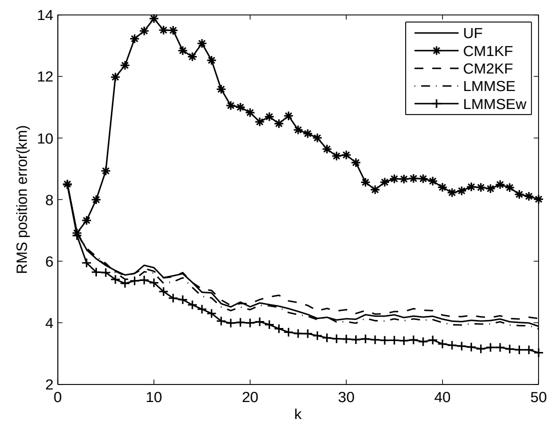

Figure 9: RMS position error comparison for case 3. Note that LMMSE is slightly better than UF.

Illustrative numerical examples have shown that in terms of credibility and accuracy, the proposed filters are preferred.

References

[1] R. K. Mehra, "A comparison of several nonlinear filters for reentry vehicle tracking," *IEEE Transactions on Automatic Control*, vol. 16, no. 4, pp. 307–319, August 1971.

[2] S. Julier, J. Uhlmann, and H. F. Durrant-Whyte, "A new method for nonlinear transformation of means and covariances in filters and estimators," *IEEE Transactions on Automatic Control*, vol. 45, no. 3, pp. 477–482, March 2000.

[3] M. Norgaard, N. K. Poulsen, and O. Ravn, "New developments in state estimation for nonlinear systems," *Automatica*, vol. 36, no. 11, pp. 1627–1638, November 2000.

[4] K. Ito and K. Q. Xiong, "Gaussian filters for nonlinear filtering problems," *IEEE Transactions on Automatic Control*, vol. 45, no. 5, pp. 910–927, May 2000.

[5] X. R. Li and V. P. Jilkov, "A survey of maneuvering target tracking - part III: Measurement models," in *Proceedings of SPIE Conference on Signal and Data Processing of Small Targets*, vol. 4473, San Diego, CA, July-August 2001, pp. 1–24.

[6] D. Lerro and Y. Bar-Shalom, "Tracking with debiased consistent converted measurements versus ekf," *IEEE Transactions on Aerospace and Electronic Systems*, vol. 29, no. 3, pp. 1015–1022, July 1993.

[7] P. Suchomski, "Explicit expressions for debiased statistics of 3d converted measurements," *IEEE Transactions on Aerospace and Electronic Systems*, vol. 35, no. 1, pp. 368–370, January 1999.

[8] S. T. Park and J. G. Lee, "Improved kalman filter design for three-dimensional radar tracking," *IEEE Transactions on Aerospace and Electronic Systems*, vol. 37, no. 2, pp. 727–739, April 2001.

[9] L. B. Mo, X. Q. Song, Y. Y. Zhou, Z. K. Sun, and Y. Bar-Shalom, "Unbiased converted measurements for tracking," *IEEE Transactions on Aerospace and Electronic Systems*, vol. 34, no. 3, pp. 1023–1027, July 1998.

[10] Z. S. Duan, C. Z. Han, and X. R. Li, "Comments on "unbiased converted measurements for tracking"," *IEEE Transac-*

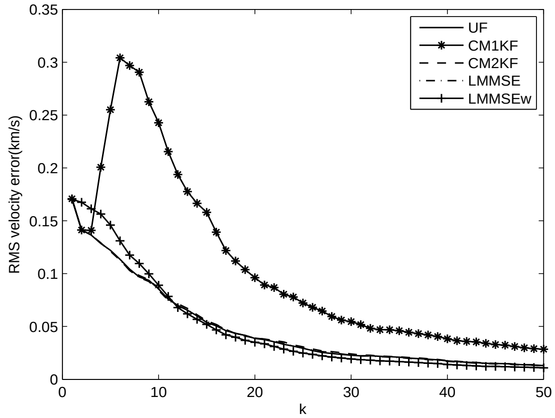

Figure 10: RMS velocity error comparison for case 3. Note that UF, CM2KF and LMMSE essentially overlap with each other.

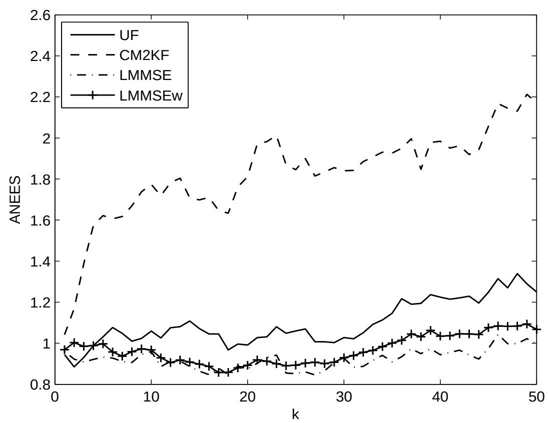

Figure 11: ANEES comparison for case 3

tions on Aerospace and Electronic Systems, vol. 40, no. 4, pp. 1374–1377, October 2004.

[11] Z. L. Zhao, X. R. Li, and V. P. Jilkov, “Best linear unbiased filtering with nonlinear measurements for target tracking,” *IEEE Transactions on Aerospace and Electronic Systems*, vol. 40, no. 4, pp. 1324–1336, October 2004.

[12] J. G. Wang, P. K. He, and T. Long, “Use of the radial velocity measurement in target tracking,” *IEEE Transactions on Aerospace and Electronic Systems*, vol. 39, no. 2, pp. 401–413, April 2003.

[13] Z. S. Duan, C. Z. Han, and X. R. Li, “Sequential nonlinear tracking filter with range-rate measurements in spherical coordinates,” in *Proceedings of the 7th International Conference on Information Fusion*, vol. 1, Stockholm, Sweden, June 2004, pp. 599–605.

[14] Y. Kosuge, H. Iwama, and Y. Miyazaki, “A tracking filter for phased array radar with range rate measurement,” in *Proceedings of 1991 International Conference on Industrial Electronics, Control and Instrumentation*, vol. 3, Kobe, October 1991, pp. 2555–2560.

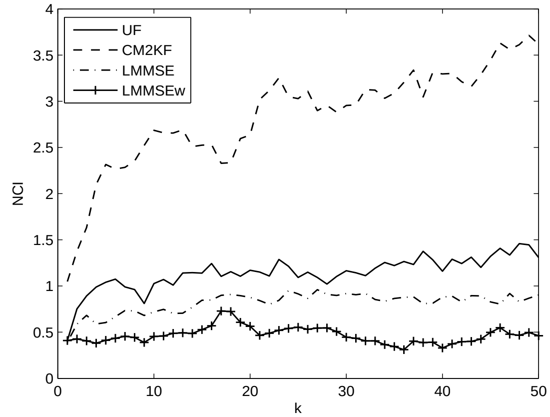

Figure 12: NCI comparison for case 3

[15] H. J. Shin, S. M. Hong, and D. H. Hong, “Adaptive-update-rate target tracking for phased-array radar,” *IEE Proceedings - Radar, Sonar and Navigation*, vol. 142, no. 3, pp. 137–143, June 1995.

[16] P. Minvielle, “Decades of improvements in re-entry ballistic vehicle tracking,” *IEEE Aerospace and Electronic Systems Magazine*, vol. 20, no. 8, pp. CF/1–CF14, August 2005.

[17] F. Daum and R. Fitzgerald, “Decoupled kalman filters for phased array radar tracking,” *IEEE Transactions on Automatic Control*, vol. 28, no. 3, pp. 269–283, March 1983.

[18] G. van Keuk, “Multihypothesis tracking with electronically scanned radar,” *IEEE Transactions on Aerospace and Electronic Systems*, vol. 31, no. 3, pp. 916–927, July 1995.

[19] S. M. Hong and Y. H. Jung, “Optimal scheduling of track updates in phased array radars,” *IEEE Transactions on Aerospace and Electronic Systems*, vol. 34, no. 3, pp. 1016–1022, July 1998.

[20] X. Tian and Y. Bar-Shalom, “Coordinate conversion and tracking for very long range radars,” *IEEE Transactions on Aerospace and Electronic Systems*, vol. 45, no. 3, pp. 1073–1088, July 2009.

[21] X. R. Li, “Recursibility and optimal linear estimation and filtering,” in *Proceedings of the 43rd IEEE Conference on Decision and Control*, Atlantis, Paradise Island, Bahamas, December 2004, pp. 1761–1766.

[22] X. R. Li and V. P. Jilkov, “A survey of maneuvering target tracking - part III: Approximation techniques for nonlinear filtering,” in *Proceedings of 2004 SPIE Conference on Signal and Data Processing of Small Targets*, vol. 5428, San Diego, CA, April 2004, pp. 537–550.

[23] ——, “Survey of maneuvering target tracking. part I: Dynamic models,” *IEEE Transactions on Aerospace and Electronic Systems*, vol. 39, no. 4, pp. 1333–1364, October 2003.

[24] X. R. Li and Z. L. Zhao, “Measuring estimator’s credibility: Noncredibility index,” in *Proceedings of the 9th International Conference on Information Fusion*, Florence, Italy, July 2006, pp. 1–8.